



Glacier National Park, Montana

Photo by Vickie Kelly, 2004

6.4 Exponential Growth and Decay

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The number of bighorn sheep in a population increases at a rate that is proportional to the number of sheep present (at least for awhile.)

So does any population of living creatures. Other things that increase or decrease at a rate proportional to the amount present include radioactive material and money in an interest-bearing account.

If the rate of change is *proportional to the amount present*, the change can be modeled by:

$$\frac{dy}{dt} = ky$$

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Rate of change is proportional to the amount present – Let's solve this differential equation.

$$\frac{1}{y} dy = k dt$$

$$\int \frac{1}{y} dy = \int k dt$$

$$y = C e^{kt}$$

SUPPOSE THAT AT $t=0$, AMOUNT PRESENT IS y_0

$$y_0 = C e^{k(0)} \Rightarrow C = y_0$$

$$y = y_0 e^{kt}$$

Exponential Change: $y = y_0 e^{kt}$

If the constant k is positive then the equation represents growth. If k is negative then the equation represents decay.



Example:

The world population in 1990 was 5.3 billion and growing 2% a year.

- Estimate the population in 2015 assuming exponential growth.
- Estimate when the population will double.

$$a) y = y_0 e^{kt}$$
$$1.02/5.3 = 5.3 e^{k(1)}$$

$$\ln(1.02) = \ln e^k$$

$$k = \ln(1.02)$$

$$y = 5.3 e^{kt}$$

$$y = 5.3 e^{k(25)}$$

$$= \boxed{8.695 \text{ Billion}}$$

$$b) y = 5.3 e^{kt}$$
$$10.6 = 5.3 e^{k(t)}$$

$$2 = e^{kt}$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k} \approx 35 \text{ yrs.}$$

IN YEAR
2025.

Doubling Time: Find a general formula for doubling time:

$$y = y_0 e^{kt}$$

$$2y_0 = y_0 e^{kt}$$

$$2 = e^{kt}$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k}$$

$$\text{Doubling Time} = \frac{\ln 2}{k}$$



$$k = \frac{\ln 2}{\text{Doubling Time}}$$

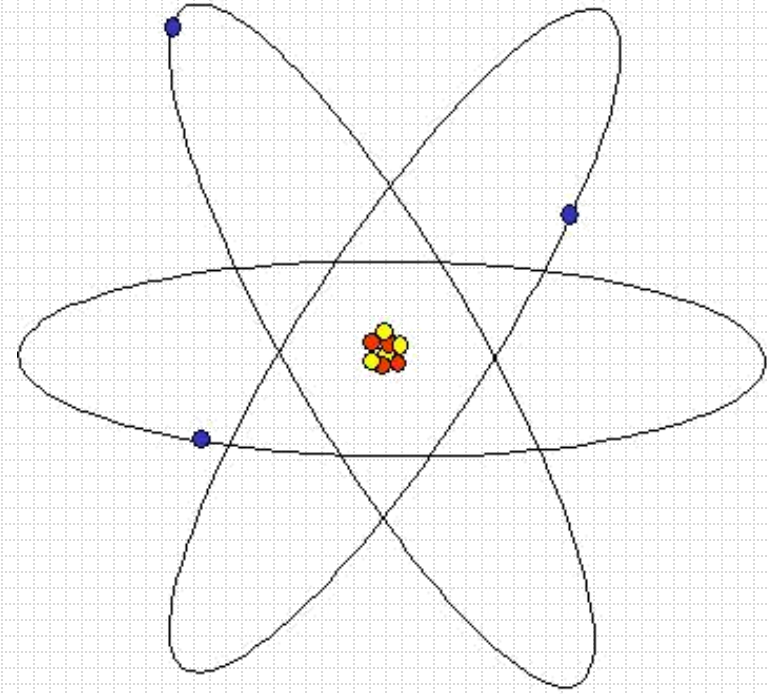
Radioactive Decay

The equation for the amount of a radioactive element left after time t is:

$$y = y_0 e^{-kt}$$

This allows the decay constant, k , to be positive.

The ***half-life*** is the time required for half the material to decay.



Half-life: Find a general formula relating half-life to k :

$$y = y_0 e^{-kt}$$

$$\frac{1}{2}y_0 = y_0 e^{-kt}$$

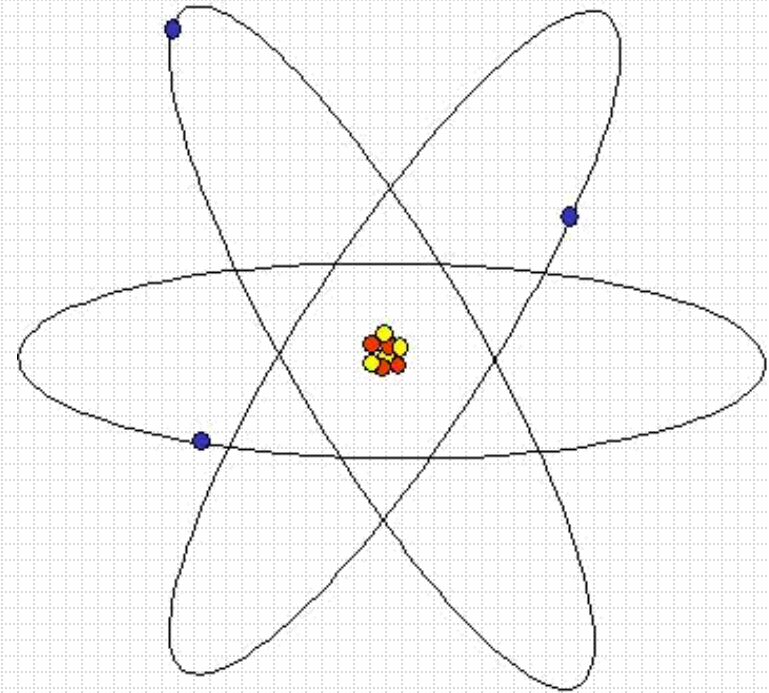
$$\ln(1/2) = -kt$$

$$t = \frac{\ln(1/2)}{-k}$$
$$t = \frac{\ln 2}{k}$$

$$HL = \frac{\ln 2}{k}$$

OR

$$k = \frac{\ln 2}{HL}$$



Suppose the half-life of a substance is 100 years. How much of a 10^{y_0} gram sample would be left after 20^t years?

$$k = \frac{\ln 2}{HL} = \frac{\ln 2}{100} \approx 0.0069$$

$$y = y_0 e^{-kt}$$

$$y_{20} = 10 e^{-k(20)}$$

$$y_{20} = \boxed{8.705 \text{ or } 8.706 \text{ g}}$$

Suppose a substance decays 30% in 12 days. What is the half-life of the substance?

$$y = y_0 e^{-kt}$$

$$70 = 100 e^{-k(12)}$$

$$0.7 = e^{-k(12)}$$

$$\ln(0.7) = -12k$$

$$k = \frac{\ln(0.7)}{-12} \Rightarrow \text{H.L.} = \frac{\ln 2}{k}$$

$$= \boxed{23.320 \text{ DAYS}}$$

Newton's Law of Cooling

Espresso left in a cup will cool to the temperature of the surrounding air. The rate of cooling is proportional to the difference in temperature between the liquid and the air.

(It is assumed that the air temperature is constant.) $\frac{dT}{dt} = -k[T - T_s]$

If we solve the differential equation:



Newton's Law of Cooling

$$T - T_s = [T_0 - T_s]e^{-kt}$$

Where T_s is the temperature of the surrounding medium, which is a constant.



Example:

$$T_s = 40$$

An egg is boiling at T_0 100° C. You take it out to cool in a T_s 5° refrigerator. After 5 minutes, the egg is 40° . How much longer will it take for the egg to reach 21° ?

$$T - T_s = [T_0 - T_s] e^{-kt}$$

$$40 - 5 = [100 - 5] e^{-k(5)}$$

$$\frac{\ln\left(\frac{35}{95}\right)}{-5} = k$$

$$T - 5 = [100 - 5] e^{-kt}$$

$$21 - 5 = 95 e^{-kt}$$

$$\frac{\ln\left(\frac{16}{95}\right)}{-k} = t$$

$$t = 8.920$$

It takes 3.920 min longer.

$$T - 5 = [40 - 5] e^{-kt}$$

$$21 - 5 = [35] e^{-kt}$$

⋮

$$t = 3.920$$

Homework:

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Section 6.4 WS - FDWK

